Development of Dynamics and Control Simulation of Large Flexible Space Systems

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This paper presents the development of an analytical formulation and general-purpose dynamics and control simulation program for large flexible multibody space systems. All of the bodies are structurally flexible and interconnected by gimbaled hinges with torsional springs and dampers which allow large-angle articulation. Control forces and torques may be applied at specified locations on the bodies or about certain gimbal axes. Dynamic behavior of these flexible interconnected bodies with time-varying boundary conditions is modeled by the quasistatic approach. System dynamics nonlinearity is linearized by the perturbation method which separates the nominal motion of an imaginative all-rigid system from the perturbed motion influenced by structural flexibilities. The direct path method can efficiently transform the individual body dynamic variables into system dynamic variables through rigid and elastokinematic relations and systematically couple the rigid-body motion with structural vibration. The linearized system dynamic equations of perturbed motion have time-varying coefficient matrices as functions of the nominal motion. Both Newtonian and Lagrangian approaches are discussed to bring out merits and drawbacks in formulation derivation. This program, named ALLFLEX, is interfaced with a structural dynamics program for modal information transfer and a generalized stability analysis program for control system design.

Introduction

FOR more than twenty years, the aerospace industries and research institutes of the world have made great progress in satellite development and space venture. In recent years, the size and topology of spacecraft systems have grown increasingly larger and more complex. With the success of Shuttle flights, the future space structures, platforms, or stations to be delivered, deployed, and assembled in space will have extremely large dimensions and very complicated configurations. The future NASA Space Operation Center, as shown in Fig. 1, is a typical example of a large flexible space system. The payloads of such a multibody system may have separate maneuver and pointing requirements, such as solar array to face the sun, antenna to point at control stations, telescope to focus at stars, and mirror reflectors for guiding laser beam to shoot at high-speed moving missile targets, etc. For reasons of weight saving, the efficient design of such systems, which pushes the limit of material strength, will result in low mass-to-size ratio, large inertias, and relatively low structural rigidity. Since many system frequencies may fall within the required control bandwidth, interaction between control dynamics and structural vibration will inevitably take place. To meet the mission requirements of these multibody systems, the designers must focus their attention on the interaction between structural vibration and system dynamics and control. Therefore, the main objectives of this research and development study are to advance the state-of-the-art in three major areas; namely, the dynamic modeling of all flexible multibody systems, the linearization of system dynamic equations, and the development of a general-purpose flexible multibody dynamics and control simulation program. With these capabilities available, we shall be in a better position to conduct the iterative design studies and improvement.

The multibody dynamics problem has been investigated by many in the past. Since Hooker and Margulies did their pioneering work on rigid multibody systems, Likins has been one of the most influential leaders in research and development of this fascinating field. Many scientists, engineers, and researchers in aerospace industries and universities have tackled this problem from different points of view using different approaches. The development of dynamic modeling and equation formulation of multibody systems can be categorized into the following stages with progressive difficulties. They are the two-rigid-body system,[1] all rigid topological tree multibody systems,[2-10] cluster system with one rigid central body and many flexible appendages, topological tree multibody systems with rigid interconnected bodies and flexible terminal bodies,[11-17] all-flexible chain system,[18,19] and all-flexible topological tree multibody systems.[20-22] The best survey of history of development on this subject can be found in Ref. 23.

In order to derive the system dynamic equations, two important tasks must be carried out carefully. The first is to transform the individual body dynamic variables into a common set of system dynamic variables. The second is to eliminate the interacting forces and torques at all hinge points. When the interconnected bodies are rigid, the kinematic relations for variable transformation are rather simple; but when they are flexible, all the relative elastic translation and rotation among hinge points in each body must be taken into consideration. Therefore, we need the elastokinematic relations to accomplish the task. From the physical point of view, the absolute pointing accuracy of the outer branch bodies are dependent upon not only the gimbal angle control but also the elastic deformation of the inner bodies in the system. The all-flexible modeling will give us more accurate simulation results than its predecessors.

Quasistatic Approach

The dynamic behavior of a typical flexible interconnected body with at least two or more hinge points is governed by partial differential equations with time-varying boundary conditions. Since the origin of body coordinates can be defined only at one of the hinge points, there will be relative elastic translation and rotation of other hinge points caused by the interacting forces and torques from adjacent bodies. A simple set of modal functions is no longer adequate to satisfy both

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the dynamic equations of equilibrium and the time-varying boundary conditions. The quasistatic approach[18,20] is very suitable to handle this type of problem. The dynamic equation of equilibrium of an elastic body is generally composed of three terms; namely, the elastic, inertial, and external forcing. When the inertial term is temporarily removed, the equation becomes quasistatic. This approach suggests that the displacements of a body be decomposed into the quasistatic and homogeneous parts. The quasistatic part will satisfy the quasistatic equation mentioned above and the time-varying boundary conditions. The homogeneous part will satisfy the homogeneous equation (which only has the elastic and inertial terms while the external forcing term is removed) and the homogeneous (zero) boundary conditions. The quasistatic field point displacement associated to the time-varying relative hinge point translation and rotation is the dominant portion of total displacement. The homogeneous field point displacement associated to fixed hinge points is merely secondary approximation and its series of solution will converge very rapidly. Both quasistatic and homogeneous modal functions (data) are standard outputs from regular structural dynamics programs.

Perturbation Method

The system dynamics in general is nonlinear because of rotational motion which creates the double omega-cross effect (i.e., double cross product of the angular velocity vector $\bar{\omega}$) and also the nonlinear trigonometric functions of gimbal and mounting angles in the coordinate transformation matrices between bodies. In order to provide better interface with the control system design, it is highly desirable to have the linearized system dynamic equations available so that stability analysis can be performed. The linearization of dynamic formulation is handled by the perturbation method, which assumes that the total motion is a superposition of nominal and perturbed motion. We first temporarily ignore all the distributed structural flexibilities in the bodies. Then the dynamics should be nominal, and the nominal motion, x^N, including nominal rigid-body translation and rotation, is governed by the nominal equation, $A^N\ddot{x}^N = F^N(x^N,\dot{x}^N,t)$. With the additional structural flexibilities, the whole system must be perturbed from its nominal position and orientation. In addition, each body also will experience dynamic elastic deformation due to quasistatic motion and natural homogeneous vibration. All of these are called the perturbed motion, x^P. Because the perturbed motion is much smaller than the nominal motion and its second-order effects can be neglected, the dynamic equations of perturbed motion, $A^P\ddot{x}^P + B^P\dot{x}^P + C^Px^P = F^P$, will be linear. However, the coefficient matrices, A^P, B^P, and C^P, are functions of time-dependent nominal motion.

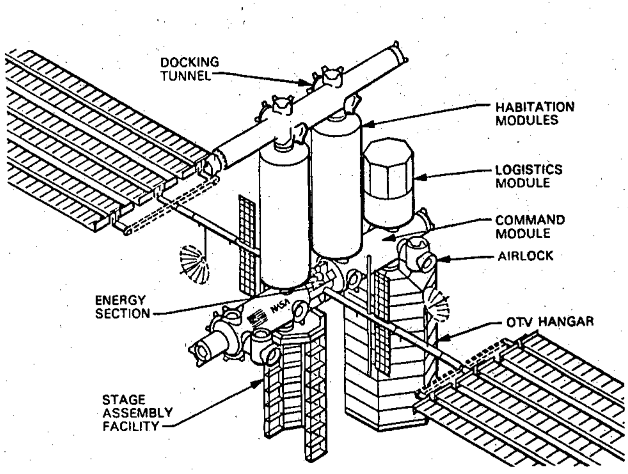

Fig. 1 Multibody system of Space Operation Center.

Direct Path Method

The direct path method[16,17] is used to develop the complex kinematic relations which will handle all possible system topologies. The direct path is a body path from the main body to a specific body which may contain certain groups of bodies in the system. Kinematic relations are based only on the interrelations among each group of bodies in the same body path, and not all the bodies in the system. In addition to the incidence matrix which uniquely defines the system topology, the limb-branch matrix is also used to define the immediate interbody relations. The combination of these two matrices can efficiently keep track of which branch body is causing what kind of quasistatic motion to its limb body and how is it interacting with the effects of other branch bodies through structural flexibility coupling.

The elimination of interacting forces and torques for a general multibody system presents difficulties to most investigators. The reasons are given as follows: Since the derivation of dynamic equations for individual bodies by either Newtonian or Lagrangian approach is rather easy, most people will naturally take this first step. But the interacting forces and torques remain in the equations. Even if the proper transformation from individual body dynamic variables to system dynamic variables by kinematic relations is performed, we still need to conduct considerable rearrangement of equations before the interacting forces and torques can be eliminated by the equal and opposite counterparts of the adjacent bodies. In this analysis, we are taking advantage of system kinematic relations by the direct path method and deriving the system dynamic equations directly. Tedious elimination of interacting forces and torques is no longer needed.

Both the Newtonian and Lagrangian approaches are investigated by the authors. Formulation by Newtonian approach is chosen to be presented in this analysis because it is more straightforward without loss of physical sense in every step of the derivation. The Lagrangian approach does have its merits because once the system kinetic energy, strain energy, and rate of change of damping energy are formed, the rest of the work will simply be partial differentiation. The drawback is that in forming the kinetic energy, we need the products of velocities which require the second-order terms of the perturbed quantities. Naturally it complicates the derivation, however, the identical final formulation can serve as independent checks for the Newtonian approach.

In summary, the direct path method enables us to go through both approaches from the system level. Its grouping capability for direct position vectors can systematically organize the contribution of individual body dynamics to system dynamics. It also enables us to derive the system dynamic equation in a rather compact form.

Multibody System Definition

The multibody system is a collection of bodies interconnected to each other by gimbals, as shown in Fig. 2. Bodies are branching out from the main body like a tree into different levels. Bodies are identified by body numbers from 1 to N. The main body in level 0 is named body 1. Bodies which are directly connected to body 1 are called the level 1 bodies and named body 2, body 3,...,body (N_1+1). Bodies which are directly connected to any level 1 bodies (but not to body 1) are called level 2 bodies and named body (N_1+2), body (N_1+3),...,body (N_1+N_2+1), etc. The numbers of bodies in various levels, $N_1, N_2, \ldots,$ are system dependent. When body 1 is directly connected to the rest of the bodies, the system is called a "cluster." When bodies are connected in series, the system is called a "chain." A general topological tree multibody system is anywhere in between these two extreme cases. Let body j be a typical "interconnected body" in level m. Then it may have only one "limb body," say, body i in level $(m-1)$; and many "branch bodies," say, body s, body t,... in level $(m+1)$. When a body has no branch body, it is called a "terminal body." The system topology may be uniquely defined by the "incidence matrix."

$\epsilon^{vj} = 1$ when body v belongs to direct path $1-j$

$= 0$ otherwise

The interbody relations are defined by the "limb-branch matrix."

$\eta^{jt} = 1$ when body t is an immediate branch body of body j

$= 0$ otherwise

Motion of a Body

The motion of each body is composed of three parts; namely, the orbital motion, the nominal motion, and the perturbed motion. They are discussed as follows: Let O_e, O_r, O_j, x^e_ξ, x^r_γ, x^j_β, e^e_ξ, e^r_γ, and e^j_β be the origins, coordinates, and unit vectors of the Earth, reference, and body j, respectively. The orbital translation is represented by the orbital position vector $\bar R^r = R^r_\xi e^e_\xi = \overline{O_e O_r}$ and it is governed by the orbital dynamic equation

$$\ddot{\bar R}^r + \zeta \bar R^r = 0 \tag{1}$$

where $\zeta = K_G M_e / |\bar R^r|^3$, K_G is the gravitational constant, and M_e is the mass of Earth. The orbital rotation is represented by the angular velocity and acceleration vectors of the reference coordinates, $\bar\omega^r = \omega^r_\gamma e^r_\gamma$ and $\dot{\bar\omega}^r = \dot\omega^r_\gamma e^r_\gamma$.

The nominal motion of the system is based on the assumption that all the distributed structural flexibilities in the bodies are ignored temporarily. Bodies will take their nominal position and orientations. The hinge point O_j between body j and its limb body, body i, is chosen to be the origin of the nominal body coordinates x^j_β. The nominal translation is represented by the nominal position vector, $\bar R^j = R^j_\gamma e^r_\gamma = \overline{O_r O_j}$ and the nominal field point P_j with respect to O_j is defined by $\bar r^j = r^j_\beta e^j_\beta = \overline{O_j P_j}$. The nominal rotation is represented by the nominal angular velocity and acceleration vectors, $\bar\omega^j = \omega^j_\beta e^j_\beta$ and $\dot{\bar\omega}^j = \dot\omega^j_\beta e^j_\beta$. It can also be represented by the nominal gimbal rotational angles θ^j_λ about the nonorthogonal gimbal rotational axes x^{jg}_λ ($\lambda = 1,2,3$) and their time derivatives. For the main body, body 1, these angles are simply the roll, pitch, and yaw Euler angles.

When the distributed structural flexibilities are taken into consideration, the positions and orientations of the bodies must be perturbed slightly from their nominal positions and orientations. The origin (also hinge point) of the instantaneous coordinates $x_\alpha^{j\prime}$ is now at O_j'. Each body also will experience elastic deformation. The perturbed motion contains the following five parts:

1) Perturbed rigid-body translation (Fig. 3a). It is defined by the perturbed rigid-body displacement vector $\Delta\bar R^j = \Delta R^j_\gamma e^r_\gamma$ which also applies to all the field points before any rigid-body rotation takes place. The instantaneous position vector from O_r to O_j' will be

$$\bar R^{j\prime} = R^{j\prime}_\gamma e^r_\gamma = \overline{O_r O_j'} = \bar R^j + \Delta\bar R^j \tag{2}$$

2) Perturbed rigid-body rotation (Fig. 3b). The instantaneous body coordinates $x^{j\prime}_\alpha$ are turned slightly from the nominal body coordinates x^j_β by the perturbed rigid-body rotational displacement vector $\Delta\bar\psi^j = \Delta\psi^j_\beta e^j_\beta$. The corresponding unit vectors are related by

$$e^{j\prime}_\alpha = (\delta_{\alpha\beta} - \Delta\tilde\psi^j_{\alpha\beta}) e^j_\beta \tag{3}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta tensor and $\Delta\tilde\psi^j_{\alpha\beta}$ is the skew-symmetric tensor of $\Delta\psi^j_\beta$. Hence, the position vector of a field point (before any elastic deformation) will be

$$\bar r^{j\prime} = r^{j\prime}_\alpha e^{j\prime}_\alpha = \bar r^j + \Delta\bar\psi^j \times \bar r^j \tag{4}$$

The perturbed rigid-body rotational displacement vector $\Delta\bar\psi^j$ is also related to the perturbed gimbal rotational angles $\Delta\bar\theta^v = \Delta\theta^v_\lambda e^{vg}_\lambda$ and the quasistatic rotation of body v [to be discussed in point 4 and Eq. (15)] which belongs to direct path $1-j$. The total gimbal rotational angles will be

$$\theta^{v\prime}_\lambda = \theta^v_\lambda + \Delta\theta^v_\lambda \qquad (\lambda = 1,2,3) \tag{5}$$

3) Quasistatic translation (Fig. 3c). All hinge points, O_j, $O_s, \ldots,$ are constrained from any relative translation or rotation with respect to each other, except O_t is allowed only the pure quasistatic translation $\bar q^{jTt} = q_\beta{}^{jTt} e^j_\beta$ with respect to O_j, but no rotation. The field point at $\bar r^j$ will be displaced quasistatically by

$$\bar u^{jTt} = \sum_{\beta=1}^{3} \bar\phi^{jTt}_\beta q^{jTt}_\beta \tag{6}$$

where $\bar\phi^{jTt}_\beta(\beta = 1,2,3)$ are called the translational quasistatic modal functions.

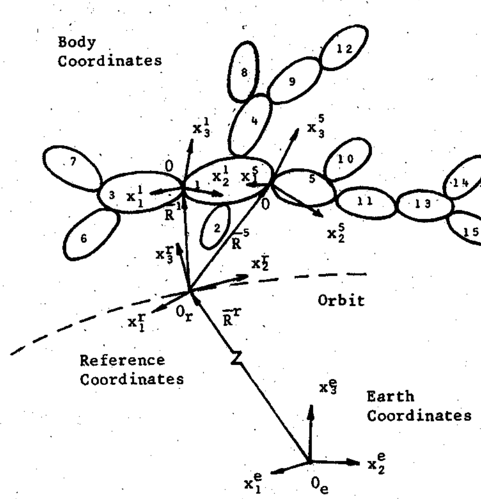

Fig. 2 Typical multibody system.

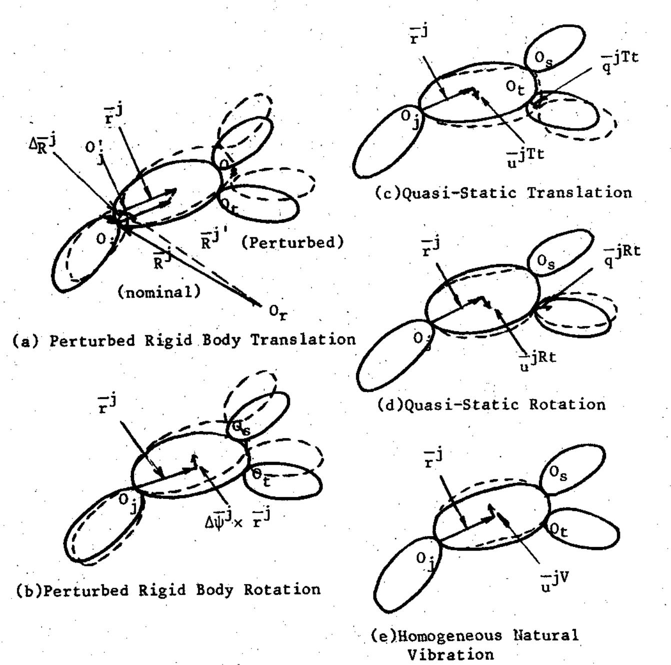

Fig. 3 Perturbed motion of a flexible interconnected body.

4) Quasistatic rotation (Fig. 3d). All hinge points, O_j, O_s,..., are constrained from any relative translation or rotation with respect to each other, except O_t is allowed only the pure quasistatic rotation $\bar{q}^{jRt} = q_\beta^{jRt} e_\beta^j$ with respect to x_β^j, but no translation. The field point at $\bar{r}^j$ will be displaced quasistatically by

$$\bar{u}^{jRt} = \sum_{\beta=1}^{3} \bar{\phi}_\beta^{jRt} q_\beta^{jRt} \tag{7}$$

where $\bar{\phi}_\beta^{jRt}$ ($\beta = 1,2,3$) are called the rotational quasistatic modal functions.

5) Homogeneous natural vibration (Fig. 3e). All hinge points, O_j, O_s, O_t, ..., are completely constrained from any relative translation or rotation with respect to each other. The body is vibrating naturally and the field point at $\bar{r}^j$ will be displaced by

$$\bar{u}^{jV} = \sum_{\ell=1}^{n_j} \bar{\phi}_\ell^{jV} q_\ell^{jV} \tag{8}$$

where $\bar{\phi}_\ell^{jV}$ ($\ell = 1,2,...,n_j$) are called the natural vibration modal functions, and q_ℓ^{jV} are the associated time functions. The choice of number of modes, n_j, for each flexible body is problem dependent.

Position Vector of a Field Point

The position vector of a field point in body j with respect to O_e of the Earth coordinates is

$$\bar{\rho}^j = \bar{R}^r + \bar{R}^j + \bar{r}^j + \Delta\bar{R}^j + \Delta\bar{\psi}^j \times \bar{r}^j + \sum_{t=j}^{N} \eta^{jt}(\bar{u}^{jTt} + \bar{u}^{jRt}) + \bar{u}^{jV} \tag{9}$$

System Kinematic Relations

The system kinematic relations are governed by the following set of direct position vectors from the nominal hinge point O_v of body v to the nominal mass center Q_j, hinge point O_j, and field point P_j of body j as shown in Fig. 4.

$$\bar{g}^{v(j)} = \sum_{p=v}^{j} \epsilon^{pj} \bar{\ell}^{p(j)} = \overline{O_v Q_j}$$

$$\bar{g}^{v(j)\Delta} = \overline{O_v O_j} = \bar{g}^{v(j)} - \bar{d}^j$$

$$\bar{g}^{v(j)*} = \overline{O_v P_j} = \bar{g}^{v(j)} - \bar{d}^j + \bar{r}^j \tag{10}$$

where

$$\bar{\ell}^{p(j)} = \ell_\beta^{p(j)} e_\beta^p = \bar{h}^{pb} = \overline{O_p O_b} \qquad (p \neq j)$$

$$\bar{\ell}^{j(j)} = \ell_\beta^{j(j)} e_\beta^j = \bar{d}^j = \overline{O_j Q_j}$$

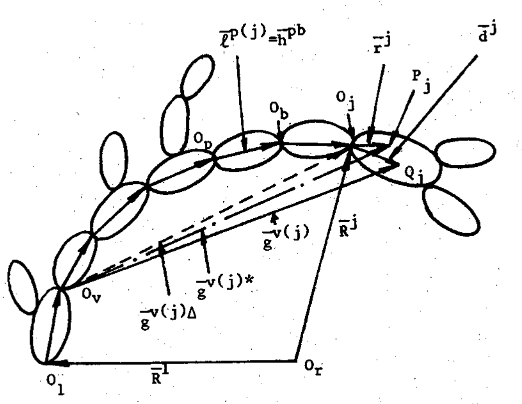

Fig. 4 Kinematic relations among bodies.

The rigid kinematic relations for the nominal variables are

$$\bar{R}^j = \bar{R}^1 + \bar{g}^{1(j)\Delta}; \qquad \bar{R}^j + \bar{r}^j = \bar{R}^1 + \bar{g}^{1(j)*} \tag{11}$$

$$\bar{\omega}^p = \sum_{v=1}^{p} \epsilon^{vp} (\delta^{v1} \bar{\omega}^r + \bar{\bar{G}}^v \cdot \dot{\bar{\theta}}^v) \tag{12}$$

where the gimbal transformation tensor is

$$\bar{\bar{G}}^v = e_\beta^v G_{\beta\mu}^v e_\mu^{vg*} \tag{13}$$

and the nonorthogonal reciprocal base vectors e_μ^{vg*} are related to the nonorthogonal base vectors e_λ^{vg} by $e_\mu^{vg*} \cdot e_\lambda^{vg} = \delta_{\mu\lambda}$. $\bar{\bar{G}}^v \cdot \dot{\bar{\theta}}^v$ will project the relative nominal gimbal angular velocity $\dot{\bar{\theta}}^v = \dot{\theta}_\lambda^v e_\lambda^{vg}$ onto the body coordinates x_β^v. The elastokinematic relations for the perturbed dynamic variables are

$$\Delta\bar{R}^j = \Delta\bar{R}^1 + \sum_{p=1}^{j} \epsilon^{pj} \Delta\bar{\psi}^p \times \bar{\ell}^{p(j)} - \Delta\bar{\psi}^j \times \bar{d}^j + \sum_{p=1}^{j} \sum_{b=p}^{j} \eta^{pb} \epsilon^{bj} \bar{q}^{pTb} \tag{14}$$

$$\Delta\bar{\psi}^p = \sum_{v=1}^{p} \epsilon^{vp} \bar{\bar{G}}^v \cdot \Delta\bar{\theta}^v + \sum_{v=1}^{p} \sum_{c=v}^{p} \eta^{vc} \epsilon^{cp} \bar{q}^{vRc} \tag{15}$$

$$\Delta\bar{R}^j + \Delta\bar{\psi}^j \times \bar{r}^j = \Delta\bar{R}^1 - \sum_{v=1}^{j} \epsilon^{vj} \bar{g}^{v(j)*} \times \bar{\bar{G}}^v \cdot \Delta\bar{\theta}^v + \sum_{v=1}^{j} \sum_{t=v}^{j} \eta^{vt} \epsilon^{tj} (\bar{q}^{vTt} - \bar{g}^{t(j)*} \times \bar{q}^{vRt}) \tag{16}$$

Position Vector of a Field Point in Terms of System Dynamic Variables

Using the above kinematic relations, the position vector is transformed into

$$\bar{\rho}^j = \bar{R}^r + \bar{R}^1 + \bar{g}^{1(j)*} + \Delta\bar{R}^1 - \sum_{v=1}^{j} \epsilon^{vj} \bar{g}^{v(j)*} \times \bar{\bar{G}}^v \cdot \Delta\bar{\theta}^v + \sum_{v=1}^{j} \sum_{t=v}^{N} \sum_{\beta=1}^{3} \eta^{vt} (\bar{\mathfrak{X}}_\beta^{vTt(j)} q_\beta^{vTt} + \bar{\mathfrak{X}}_\beta^{vRt(j)} q_\beta^{vRt}) + \sum_{v=1}^{j} \sum_{\ell=1}^{n_v} \bar{\mathfrak{X}}_\ell^{vV(j)} q_\ell^{vV} \tag{17}$$

where the body j's contributions to the system modal functions are

$$\bar{\mathfrak{X}}_\beta^{vTt(j)} = \delta^{vj} \bar{\phi}_\beta^{jTt} + \epsilon^{tj} e_\beta^v$$

$$\bar{\mathfrak{X}}_\beta^{vRt(j)} = \delta^{vj} \bar{\phi}_\beta^{jRt} - \epsilon^{tj} \bar{g}^{t(j)*} \times e_\beta^v$$

$$\bar{\mathfrak{X}}_\ell^{vV(j)} = \delta^{vj} \bar{\phi}_\ell^{jV} \tag{18}$$

Acceleration Vector of a Field Point

The acceleration of a field point in body j will be

$$\bar{a}^j = \frac{d^2 \bar{\rho}^j}{dt^2} = \bar{\rho}^{j\infty} \tag{19}$$

where the total time derivatives are rather straightforward.

Environmental Disturbances

There are many environmental disturbances to the space system, such as gravitation, solar radiation, aerodynamic drag, magnetic dipole moment interaction, etc. We shall only present the gravitational effect where the disturbing force vector on unit mass of body j is

$$\bar{f}^{jG} = -\zeta\Big\{\bar{R}^r + \bar{\bar{\Omega}}\cdot\Big[\bar{R}^1 + \bar{g}^{1(j)*} + \Delta\bar{R}^1 - \sum_{v=1}^{j}\epsilon^{vj}\bar{g}^{v(j)*} \times \bar{\bar{G}}^v\cdot\Delta\bar{\theta}^v + \sum_{v=1}^{j}\sum_{t=v}^{N}\sum_{\beta=1}^{3}\eta^{vt}(\mathfrak{X}_\beta^{vTt(j)}q_\beta^{vTt} + \mathfrak{X}_\beta^{vRt(j)}q_\beta^{vRt}) + \sum_{v=1}^{j}\sum_{\ell=1}^{n_v}\mathfrak{X}_\ell^{vV(j)}q_\ell^{vV}\Big]\Big\} \tag{20}$$

where $\bar{\bar{\Omega}} = \bar{\bar{\delta}} - 3ee$ and $e = \bar{R}^r/|\bar{R}^r|$ = unit vector along $\bar{R}^r$.

Control Forces and Torques

The control forces at the specified locations $\bar{r}^{j(i)}$ ($i = 1,2,\ldots,N_j$) in body j and the control torques about gimbal axes x_μ^{ug} of body u are decomposed into nominal and perturbed portions as follows:

$$\bar{F}^{jc(i)\prime} = \bar{F}^{jc(i)} + \Delta\bar{F}^{jc(i)} \qquad T_\mu^{ugc\prime} = T_\mu^{ugc} + \Delta T_\mu^{ugc} \tag{21}$$

They may be in the form of reaction jets, control moment gyros, or simply torquers.

System Dynamic Equations

Derivations of system dynamic equations are based on Newtonian approach which is to balance forces, torques about various gimbal axes, work done, and energy dissipated and stored that are associated to various quasistatic motion or natural vibration. They take the following forms:

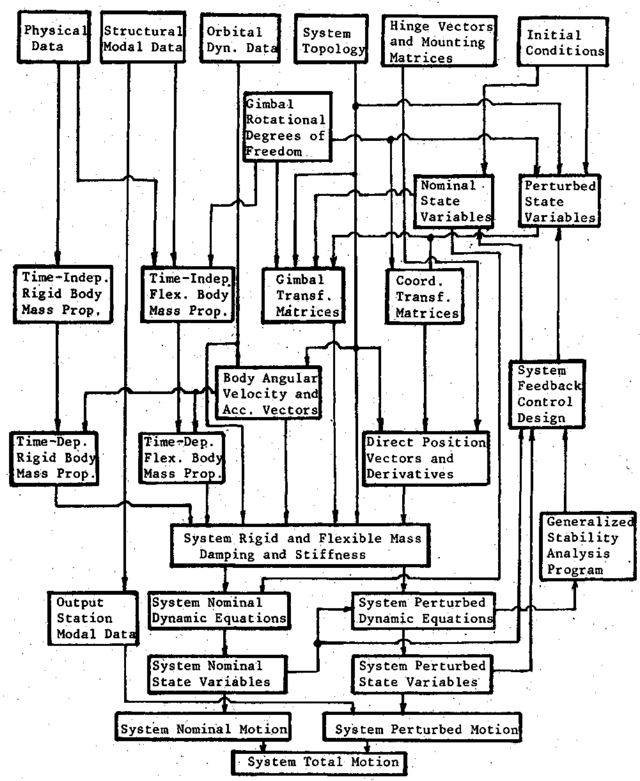

Fig. 5 ALLFLEX multibody dynamics and control simulation computer program.

$$e_\sigma^r\cdot\sum_{j=1}^{N}\int(\bar{a}^j - \bar{f}^{jG})\mathrm{d}m^j = e_\sigma^r\cdot\bar{F}^{c\prime} \qquad (\sigma = 1,2,3) \tag{22a}$$

$$e_\mu^{ug}\cdot\bar{\bar{G}}^{u(T)}\cdot\sum_{j=u}^{N}\epsilon^{uj}\int\bar{L}^{u(j)}\times(\bar{a}^j - \bar{f}^{jG})\mathrm{d}m^j + V_\mu^{ug}(\dot{\theta}_\mu^u + \Delta\dot{\theta}_\mu^u) + K_\mu^{ug}(\theta_\mu^u + \Delta\theta_\mu^u) = e_\mu^{ug}\cdot\bar{\bar{G}}^{u(T)}\cdot\bar{T}^{uc\prime} + T_\mu^{ugc\prime}$$

$$(u = 1,2,\ldots N;\ \mu = 1,2,3) \tag{22b}$$

$$\eta^{us}\sum_{j=u}^{N}\epsilon^{uj}\Big\{\int(\mathfrak{X}_\alpha^{uTs(j)} + \Delta\bar{\psi}^u\times\mathfrak{X}_\alpha^{uTs(j)})\cdot(\bar{a}^j - \bar{f}^{jG})\mathrm{d}m^j + \sum_{v=1}^{N}\sum_{t=v}^{N}\sum_{\beta=1}^{3}\delta^{uv}m^v\eta^{vt}(K_{\alpha\beta}^{vTTst}q_\beta^{vTt} + V_{\alpha\beta}^{vTTst}\dot{q}_\beta^{vTt} + K_{\alpha\beta}^{vTRst}q_\beta^{vRt} + V_{\alpha\beta}^{vTRst}\dot{q}_\beta^{vRt})\Big\} = \eta^{us}\mathfrak{W}_\alpha^{uTs}$$

$$(u = 1,2,\ldots N;\quad s = u,\ldots N;\quad \alpha = 1,2,3) \tag{22c}$$

$$\eta^{us}\sum_{j=u}^{N}\epsilon^{uj}\Big\{\int(\mathfrak{X}_\alpha^{uRs(j)} + \Delta\bar{\psi}^u\times\mathfrak{X}_\alpha^{uRs(j)})\cdot(\bar{a}^j - \bar{f}^{jG})\mathrm{d}m^j + \sum_{v=1}^{N}\sum_{t=v}^{N}\sum_{\beta=1}^{3}\delta^{uv}m^v\eta^{vt}(K_{\alpha\beta}^{vRTst}q_\beta^{vTt} + V_{\alpha\beta}^{vRTst}\dot{q}_\beta^{vTt} + K_{\alpha\beta}^{vRRst}q_\beta^{vRt} + V_{\alpha\beta}^{vRRst}\dot{q}_\beta^{vRt})\Big\} = \eta^{us}\mathfrak{W}_\alpha^{uRt}$$

$$(u = 1,2,\ldots N;\quad s = u,\ldots N;\quad \alpha = 1,2,3) \tag{22d}$$

$$\sum_{j=u}^{N}\epsilon^{uj}\Big\{\int(\mathfrak{X}_k^{uV(j)} + \Delta\bar{\psi}^u\times\mathfrak{X}_k^{uV(j)})\cdot(\bar{a}^j - \bar{f}^{jG})\mathrm{d}m^j + \sum_{v=u}^{N}\sum_{\ell=1}^{n_v}\delta^{uv}m^v(K_{k\ell}^{vVV}q_\ell^{vV} + V_{k\ell}^{vVV}\dot{q}_\ell^{vV})\Big\} = \mathfrak{W}_k^{uV}$$

$$(u = 1,2,\ldots N;\quad k = 1,2,\ldots n_u) \tag{22e}$$

where

$$\bar{F}^{c\prime} = \sum_{j=1}^{N}\sum_{i=1}^{N_j}\bar{F}^{jc(i)\prime} = \bar{F}^c + \Delta\bar{F}^c, \qquad \bar{T}^{uc\prime} = \sum_{j=u}^{N}\epsilon^{uj}\sum_{i=1}^{N_j}\bar{L}^{u(ji)}\times\bar{F}^{jc(i)\prime} = \bar{T}^{uc} + \Delta\bar{T}^{uc}$$

$$\mathfrak{W}_\alpha^{uTs} = \sum_{j=u}^{N}\epsilon^{uj}\sum_{i=1}^{N_j}\mathfrak{X}_\alpha^{uTs(ji)}\cdot\bar{F}^{jc(i)\prime}, \qquad \mathfrak{W}_\alpha^{uRs} = \sum_{j=u}^{N}\epsilon^{uj}\sum_{i=1}^{N_j}\mathfrak{X}_\alpha^{uRs(ji)}\cdot\bar{F}^{jc(i)\prime}, \qquad \mathfrak{W}_k^{uV} = \sum_{j=u}^{N}\epsilon^{uj}\sum_{i=1}^{N_j}\mathfrak{X}_k^{uV(ji)}\cdot\bar{F}^{jc(i)\prime} \tag{23}$$

$$\bar{L}^{u(j)} = \bar{g}^{u(j)*} - \sum_{v=u}^{j}\epsilon^{vj}\bar{g}^{v(j)*}\times\bar{\bar{G}}^v\cdot\Delta\bar{\theta}^v + \sum_{v=u}^{j}\sum_{t=v}^{N}\sum_{\beta=1}^{3}\eta^{vt}(\mathfrak{X}_\beta^{vTt(j)}q_\beta^{vTt} + \mathfrak{X}_\beta^{vRt(j)}q_\beta^{vRt}) + \sum_{v=u}^{j}\sum_{\ell=1}^{n_v}(\mathfrak{X}_\ell^{vV(j)}q_\ell^{vV}) \tag{24}$$

$$\bar L^{u(ji)} = \bar L^{u(j)}, \quad \mathfrak{X}_\alpha^{uTs(ji)} = \mathfrak{X}_\alpha^{uTs(j)}, \quad \mathfrak{X}_\alpha^{uRs(ji)} = [\mathfrak{X}_\alpha^{uRs(j)}, \quad \mathfrak{X}_\ell^{uV(ji)} = \mathfrak{X}_\ell^{uV(j)} \quad \text{when } \bar r^j = \bar r^{j(i)} \tag{25}$$

K_μ^{ug}, V_μ^{ug} = torsional spring and damping constants about gimbal axes x_μ^{ug}

$K_{\alpha\beta}^{vXYst}; V_{\alpha\beta}^{vXYst}$ = generalized stiffness and damping matrices of body v for quasistatic motion ($X, Y = T,R$)

$K_{\ell\ell}^{vVV}; V_{\ell\ell}^{vVV}$ = generalized stiffness and damping matrices of body v for natural vibration

$\bar{\bar G}^{u(T)} = e_\mu^{ug\,*} G_{\mu\alpha}^{u(T)} e_\alpha^u$ = transpose of $\bar{\bar G}^u$

System Dynamic Equations of Nominal Motion

The system dynamic equations of nominal motion are:

$$m_T \begin{bmatrix} e_\sigma^r \cdot e_\gamma^r & -e_\sigma^r \cdot \bar D^v \times \bar{\bar G}^v \cdot e_\lambda^{vg} \\ \hline e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot \bar D^u \times e_\gamma^r & e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot \bar{\bar I}^{uv} \cdot \bar{\bar G}^v \cdot e_\lambda^{vg} \end{bmatrix} \begin{Bmatrix} \ddot R_\gamma^l \\ \hline \ddot\theta_\lambda^v \end{Bmatrix}$$

$$+ m_T \begin{bmatrix} 2e_\sigma^r \cdot (\bar\omega^r \times e_\gamma^r) & -e_\sigma^r \cdot (\bar D^v \times \bar{\bar G}^{v0} + \bar D^{v0} + \bar{\bar G}^v) \cdot e_\lambda^{vg} \\ \hline 2e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot \bar D^u \times (\bar\omega^r \times e_\gamma^r) & e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot (\bar{\bar I}^{uv} \cdot \bar{\bar G}^{v0} + \bar{\bar J}^{uv} \cdot \bar{\bar G}^v) \cdot e_\lambda^{vg} + (1/m_T)\delta^{uv}\delta_{\mu\lambda} V_\lambda^{vg} \end{bmatrix} \begin{Bmatrix} \dot R_\gamma^l \\ \hline \dot\theta_\lambda^v \end{Bmatrix}$$

$$+ m_T \begin{bmatrix} e_\sigma^r \cdot (\bar{\bar\Lambda}^r + \zeta\,\bar{\bar\Omega}) \cdot e_\gamma^r & 0 \\ \hline e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot \bar D^u \times (\bar{\bar\Lambda}^r + \zeta\,\bar{\bar\Omega}) \cdot e_\gamma^r & (1/m_T)\delta^{uv}\delta_{\mu\lambda} K_\lambda^{vg} \end{bmatrix} \begin{Bmatrix} R_\gamma^l \\ \hline \theta_\lambda^v \end{Bmatrix}$$

$$= -m_T \begin{Bmatrix} e_\sigma^r \cdot (-\bar D^l \times \dot{\bar\omega}^r - \bar D^{l0} \times \bar\omega^r + \zeta\,\bar{\bar\Omega} \cdot \bar D^l) \\ \hline e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot (\bar{\bar I}^{ul} \cdot \dot{\bar\omega}^r + \bar{\bar J}^{ul} \cdot \bar\omega^r + \zeta \bar{\bar H}^{ul(G)}) \end{Bmatrix} + \begin{Bmatrix} e_\sigma^r \cdot \bar F^c \\ \hline e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot \bar T^{uc} + T_\mu^{ugc} \end{Bmatrix}$$

$$(\sigma = 1,2,3;\ u = 1,2,\ldots,N;\ \mu = 1,2,3)\ (\text{sum on } \gamma = 1,2,3;\ v = 1,2,\ldots,N;\ \lambda = 1,2,3) \tag{26}$$

where $\bar{\bar\Lambda}^r \cdot = \dot{\bar\omega}^r \times + \bar\omega^r \times \bar\omega^r \times$, and superscript 0 means the total time derivative. Since the orbital dynamic equation, Eq. (1), is implicitly included in $(\bar a^j - \bar f^{jG})$, this part of equilibrium separated immediately. Furthermore, the system dynamic equations of nominal motion are included in Eqs. (22a) and (22b). They can also be separated easily from the system dynamic equations of perturbed motion.

System Dynamic Equations of Perturbed Motion

The system dynamic equations of perturbed motion are:

$$m_T \begin{bmatrix} e_\sigma^r \cdot e_\gamma^r & -e_\sigma^r \cdot \bar D^v \times \bar{\bar G}^v \cdot e_\lambda^{vg} & e_\sigma^r \cdot \bar\Psi_n^{l-vYt} \\ \hline e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot \bar D^u \times e_\gamma^r & e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot \bar{\bar I}^{uv} \cdot \bar{\bar G}^v \cdot e_\lambda^{vg} & e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot \bar Y_n^{u-vYt} \\ \hline \bar\Psi_m^{uXs-l\,*} \cdot e_\gamma^r & \bar Y_m^{uXs-v\,*} \cdot \bar{\bar G}^v \cdot e_\lambda^{vg} & M_{mn}^{uXs-vYt} \end{bmatrix} \begin{Bmatrix} \Delta\ddot R_\gamma^l \\ \hline \Delta\ddot\theta_\lambda^v \\ \hline \ddot z_n^{vYt} \end{Bmatrix}$$

$$+2m_T \begin{bmatrix} e_\sigma^r \cdot (\bar\omega^r \times e_\gamma^r) & -e_\sigma^r \cdot (\bar D^v \times \bar{\bar G}^v)^0 \cdot e_\lambda^{vg} & e_\sigma^r \cdot \bar\Psi_n^{l-vYt0} \\ \hline e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot \bar D^u \times (\bar\omega^r \times e_\gamma^r) & e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot (\bar{\bar I}^{uv} \cdot \bar{\bar G}^{v0} + \bar{\bar J}^{uv} \cdot \bar{\bar G}^v) \cdot e_\lambda^{vg} + (1/m_T)\delta^{uv}\delta_{\mu\lambda} V_\lambda^{vg} & e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot \bar B_n^{u-vYt} \\ \hline \bar\Psi_m^{uXs-l\,*} \cdot (\bar\omega^r \times e_\gamma^r) & (\bar Y_m^{uXs-v\,*} \cdot \bar{\bar G}^{v0} + \bar B_m^{uXs-v\,*} \cdot \bar{\bar G}^v) \cdot e_\lambda^{vg} & C_{mn}^{uXs-vYt} + \tfrac{1}{2}\delta^{uv}\mu^v V_{mn}^{vXYst} \end{bmatrix} \begin{Bmatrix} \Delta\dot R_\gamma^l \\ \hline \Delta\dot\theta_\lambda^v \\ \hline \dot z_n^{vYt} \end{Bmatrix}$$

$$+m_T \begin{bmatrix} e_\sigma^r \cdot (\bar{\bar\Lambda}^r + \zeta\bar{\bar\Omega}) \cdot e_\gamma^r & -e_\sigma^r \cdot [(\bar D^v \times \bar{\bar G}^v)^{\infty} + \zeta\bar{\bar\Omega} \cdot \bar D^v .\ \bar{\bar G}^v] \cdot e_\lambda^{vg} & e_\sigma^r \cdot (\bar\Psi_n^{l-vYt\infty} + \zeta\,\bar{\bar\Omega} \cdot \bar\Psi_n^{l-vYt}) \\ \hline e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot \bar D^u \times (\bar{\bar\Lambda}^r + \zeta\bar{\bar\Omega}) \cdot e_\gamma^r & e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot [\bar{\bar I}^{uv} \cdot \bar{\bar G}^{v\infty} + 2\bar{\bar J}^{uv} \cdot \bar{\bar G}^{v0} + (\bar{\bar S}^{uv} + \xi\bar{\bar S}^{uv\,\prime}) \cdot \bar{\bar G}^v] \cdot e_\lambda^{vg} + (1/m_T)\delta^{uv}\delta_{\mu\lambda} K_\lambda^{vg} & e_\mu^{ug} \cdot \bar{\bar G}^{u(T)} \cdot (\bar P_n^{u-vYt} + \zeta\bar P_n^{u-vYt\,\prime}) \\ \hline \bar\Psi_m^{uXs-l\,*} \cdot (\bar{\bar\Lambda}^r + \zeta\bar{\bar\Omega}) \cdot e_\gamma^r & [\bar Y_m^{uXs-v\,*} \cdot \bar{\bar G}^{v\infty} + 2\bar B_m^{uXs-v\,*} \cdot \bar{\bar G}^{v0} + (\bar P_m^{uXs-v\,*} + \zeta\bar P_m^{uXs-v\,*\,\prime}) \cdot \bar{\bar G}^v] \cdot e_\lambda^{vg} & Q_{mn}^{uXs-vYt} + \zeta Q_{mn}^{uXs-vYt\,\prime} + \delta^{uv}\mu^v K_{mn}^{vXYst} \end{bmatrix} \begin{Bmatrix} \Delta R_\gamma^l \\ \hline \Delta\theta_\lambda^v \\ \hline z_n^{vYt} \end{Bmatrix}$$

$$
= -m_T \left\{ \begin{array}{c} 0 \\ \hline 0 \\ \hline \bar{\Psi}_m^{uXs-1*} \cdot (\bar{R}^{1\infty} + \zeta \bar{\bar{\Omega}} \cdot \bar{R}^1) + \bar{Y}_m^{uXs-v*} \cdot [\delta^{v1} \dot{\bar{\omega}}^r + (\bar{\bar{G}}^v \cdot \dot{\bar{\theta}}^v)^0] \\ + \bar{B}_m^{uXs-v*} \cdot [\delta^{v1} \bar{\omega}^r + \bar{\bar{G}}^v \dot{\bar{\theta}}^v] + \zeta U_m^{uXs(G)} \end{array} \right\} + \left\{ \begin{array}{c} e_\sigma^r \cdot \Delta \bar{F}^c \\ \hline e_\mu^{ug} \cdot \bar{\bar{G}}^{u(T)} \cdot \Delta \bar{T}^{uc} + \Delta T_\mu^{ugc} \\ \hline X_m^{uXs} \end{array} \right\}
$$

($\sigma = 1,2,3$; $u = 1,2,\ldots,N$; $\mu = 1,2,3$
when $X = T$, $R \to s = u,\ldots,N$ and $m \to \alpha = 1,2,3$
when $X = V \to s$ vanishes and $m \to k = 1,2,\ldots,n_u$)

(sum on $\gamma = 1,2,3$; $v = 1,2,\ldots,N$; $\lambda = 1,2,3$
when $Y = T$, $R \to t = v,\ldots,N$ and $n \to \beta = 1,2,3$
when $Y = V \to t$ vanishes and $n \to \ell = 1,2,\ldots,n_v$) (27)

and

$$
z_n^{vYt} = \left\{ \begin{array}{c} \eta^{vt} q_\beta^{vTt} \\ \hline \eta^{vt} q_\beta^{vRt} \\ \hline q_\ell^{vV} \end{array} \right\} \tag{28}
$$

Individual Body Properties

The time-independent individual body properties are:

$$
\begin{array}{lll}
m^j = \int dm^j & M_{mn}^{jXYst} = (1/m^j) \int \bar{\phi}_m^{jXs} \cdot \bar{\phi}_n^{jYt} dm^j & \bar{\bar{w}}_{mn}^{jXYst} = (1/m^j) \int \bar{\phi}_m^{jXs} \bar{\phi}_n^{jYt} dm^j \\
\bar{d}^j = (1/m^j) \int \bar{r}^j dm^j & \bar{Y}_n^{jYt} = (1/m^j) \int \bar{r}^j \times \bar{\phi}_n^{jYt} dm^j & \bar{\bar{I}}^j = p^j \bar{\bar{\delta}} - \bar{\bar{A}}^j \\
\bar{\Phi}_n^{jYt} = (1/m^j) \int \bar{\phi}_n^{jYt} dm^j & \bar{Z}_{mn}^{jXYst} = (1/m^j) \int \bar{\phi}_m^{jXs} \times \bar{\phi}_n^{jYt} dm^j & \bar{\bar{N}}_n^{jYt} = b_n^{jYt} \bar{\bar{\delta}} - \bar{\bar{c}}_n^{jYt} \\
p^j = (1/m^j) \int \bar{r}^j \cdot \bar{r}^j dm^j & \bar{\bar{A}}^j = (1/m^j) \int \bar{r}^j \bar{r}^j dm^j & \bar{\bar{N}}_m^{jXs*} = b_m^{jXs*} \bar{\bar{\delta}} - \bar{\bar{c}}_m^{jXs*} \\
b_n^{jYt} = (1/m^j) \int \bar{r}^j \cdot \bar{\phi}_n^{jYt} dm^j & \bar{\bar{c}}_n^{jYt} = (1/m^j) \int \bar{r}^j \bar{\phi}_n^{jYt} dm^j & \bar{\bar{E}}_{mn}^{jXYst} = M_{mn}^{jXYst} \bar{\bar{\delta}} - \bar{\bar{w}}_{mn}^{jXYst} \\
b_m^{jXs*} = (1/m^j) \int \bar{\phi}_m^{jXs} \cdot \bar{r}^j dm^j & \bar{\bar{c}}_m^{jXs*} = (1/m^j) \int \bar{\phi}_m^{jXs} \bar{r}^j dm^j &
\end{array} \tag{29}
$$

and the time-dependent individual body properties influenced by nominal motion are

$$
\begin{array}{ll}
\bar{\bar{J}}^j = -\bar{\omega}^j \times \bar{\bar{A}}^j & \bar{B}_m^{jXs*} = -\bar{\omega}^j \cdot \bar{\bar{N}}_m^{jXs*} \\
\bar{\bar{S}}^j = -\dot{\bar{\omega}}^j \times \bar{\bar{A}}^j - \bar{\bar{A}}^j \times \dot{\bar{\omega}}^j & \bar{P}_n^{jYt} = (\bar{\bar{N}}_n^{jYt} + \bar{\bar{N}}_n^{jYt*}) \cdot \dot{\bar{\omega}}^j + \bar{\omega}^j \times (\bar{\bar{N}}_n^{jYt} + \bar{\bar{N}}_n^{jYt*}) \cdot \bar{\omega}^j - \bar{R}^{j\infty} \times \bar{\Phi}_n^{jYt} \\
\quad + \bar{\bar{A}}^j \cdot (\bar{\omega}^j \bar{\omega}^j) - (\bar{\omega}^j \bar{\omega}^j) \cdot \bar{\bar{A}}^j + \bar{R}^{j\infty} \times (\bar{d}^j \times & \bar{P}_m^{jXs*} = \bar{\Phi}_m^{jXs} \times \bar{R}^{j\infty} \\
\bar{\Phi}_n^{jYt0} = \bar{\omega}^j \times \bar{\Phi}_n^{jYt} & C_{mn}^{jXYst} = -\bar{Z}_{mn}^{jXYst} \cdot \bar{\omega}^j \\
\bar{\Phi}_n^{jYt\infty} = \bar{\bar{\Lambda}}^j \cdot \bar{\Phi}_n^{jYt} & Q_{mn}^{jXYst} = -\bar{Z}_{mn}^{jXYst} \cdot \dot{\bar{\omega}}^j - \bar{\omega}^j \cdot \bar{\bar{E}}_{mn}^{jXYst} \cdot \bar{\omega}^j \\
\bar{B}_n^{jYt} = \bar{\bar{N}}_n^{jYt*} \cdot \bar{\omega}^j &
\end{array} \tag{30}
$$

System Properties

The rigid system properties are

$$
m_T = \sum_{j=1}^{N} m^j, \qquad \mu^j = m^j / m_T, \qquad \gamma^{st} = \sum_{j=s,t}^{N} \epsilon^{sj} \epsilon^{tj} \mu^j
$$

$$
\bar{D}^{st} = \sum_{j=s,t}^{N} \epsilon^{sj} \epsilon^{tj} \mu^j \bar{g}^{t(j)} \quad (\bar{D}^{1t} = \bar{D}^t), \qquad \bar{D}^{st*} = \sum_{j=s,t}^{N} \epsilon^{sj} \epsilon^{tj} \mu^j \bar{g}^{s(j)} \quad (\bar{D}^{s1} = \bar{D}^s)
$$

$$
\bar{\bar{I}}^{uv} \cdot = \sum_{j=u,v}^{N} \epsilon^{uj} \epsilon^{vj} \mu^j [\bar{\bar{I}}^j \cdot + \bar{d}^j \times (\bar{d}^j \times - \bar{g}^{u(j)} \times (\bar{g}^{v(j)}) \times]
$$

$$
\bar{\bar{J}}^{uv} \cdot = \sum_{j=u,v}^{N} \epsilon^{uj} \epsilon^{vj} \mu^j [\bar{\bar{J}}^j \cdot + \bar{d}^j \times (\bar{d}^{j0} \times - \bar{g}^{u(j)} + (\bar{g}^{v(j)0} \times]
$$

$$
\bar{\bar{S}}^{uv} \cdot = \sum_{j=u,v}^{N} \epsilon^{uj} \epsilon^{vj} \mu^j [\bar{\bar{S}}^j \cdot + \bar{d}^j \times (\bar{d}^{j\infty} \times - \bar{g}^{u(j)} \times (\bar{g}^{v(j)\infty}) \times + (\bar{R}^{j\infty} + \bar{d}^{j\infty}) \times (\bar{g}^{v(j)\Delta})] \tag{31}
$$

The rigid-flexible coupling system properties are

$$
\left\{\begin{array}{c}
\bar{\Psi}_n^{u-vYt} \\ \hline \bar{Y}_n^{u-vYt} \\ \hline \bar{B}_n^{u-vYt} \\ \hline \bar{P}_n^{u-vYt}
\end{array}\right\}
=\epsilon^{uv}\mu^v
\left\{\begin{array}{c}
\bar{\Phi}_n^{vYt} \\ \hline \bar{Y}_n^{vYt}+\bar{g}^{u(v)\Delta}\times\bar{\Phi}_n^{vYt} \\ \hline \bar{B}_n^{vYt}+\bar{g}^{u(v)\Delta}\times\bar{\Phi}_n^{vYt0} \\ \hline \bar{P}_n^{vYt}+\bar{g}^{u(v)\Delta}\times\bar{\Phi}_n^{vYt\infty}
\end{array}\right\}
+\left\{\begin{array}{c:c:c}
\gamma^{ut}e_\beta^v & -\bar{D}^{ut}\times e_\beta^v & 0 \\ \hline
\bar{D}^{ut*}\times e_\beta^v & \bar{\bar{I}}^{ut}\cdot e_\beta^v & 0 \\ \hline
\bar{D}^{ut*}\times e_\beta^{v0} & \bar{\bar{I}}^{ut}\cdot e_\beta^{v0}+\bar{\bar{J}}^{ut}\cdot e_\beta^v & 0 \\ \hline
\bar{D}^{ut*}\times e_\beta^{v\infty}+\bar{W}^{ut*}\times e_\beta^v & \bar{\bar{I}}^{ut}\cdot e_\beta^{v\infty}+2\bar{\bar{J}}^{ut}\cdot e_\beta^{v0}+\bar{\bar{S}}^{ut}\cdot e_\beta^v & 0
\end{array}\right\}
$$

$$
\qquad\qquad (Y=T) \qquad\qquad\qquad (Y=R) \qquad\qquad\qquad (Y=V) \qquad (32)
$$

$$
\left\{\begin{array}{c}
\bar{\Psi}_m^{uXs-v*} \\ \hline \bar{Y}_m^{uXs-v*} \\ \hline \bar{B}_m^{uXs-v*} \\ \hline \bar{P}_m^{uXs-v*}
\end{array}\right\}
=\epsilon^{vu}\mu^u
\left\{\begin{array}{c}
\bar{\Phi}_m^{uXs} \\ \hline \bar{Y}_m^{uXs}-\bar{\Phi}_m^{uXs}\times\bar{g}^{v(u)\Delta} \\ \hline \bar{B}_m^{uXs*}-\bar{\Phi}_m^{uXs}\times\bar{g}^{v(u)\Delta 0} \\ \hline \bar{P}_m^{uXs*}-\bar{\Phi}_m^{uXs}\times\bar{g}^{v(u)\Delta\infty}
\end{array}\right\}
+\left\{\begin{array}{c:c:c}
e_\alpha^u\gamma^{sv} & -e_\alpha^u\times\bar{D}^{sv*} & 0 \\ \hline
e_\alpha^u\times\bar{D}^{sv} & e_\alpha^u\cdot\bar{\bar{I}}^{sv} & 0 \\ \hline
e_\alpha^u\times\bar{D}^{sv0} & e_\alpha^u\cdot\bar{\bar{J}}^{sv} & 0 \\ \hline
e_\alpha^u\cdot\bar{D}^{sv\infty}+e_\alpha^u\cdot\bar{W}^{sv} & e_\alpha^u\cdot\bar{\bar{S}}^{sv} & 0
\end{array}\right\}
$$

$$
\qquad\qquad (X=T) \qquad\qquad\qquad (X=R) \qquad\qquad\qquad (X=V) \qquad (33)
$$

where $\bar{W}^{ut*}=\gamma^{ut}\bar{R}^{u\infty}+\bar{D}^{ut*\infty}$ and $\bar{W}^{sv}=\gamma^{sv}\bar{R}^{v\infty}+\bar{D}^{sv\infty}$.

The flexible system properties are

$$
M_{mn}^{uXs-vYt}=\left[\begin{array}{c:c:c}
\begin{array}{c}\delta^{uv}\mu^v M_{\alpha\beta}^{vTTst}\\ +e_\alpha^u\cdot\bar{\Psi}_\beta^{s-vTt}\\ +\bar{\Psi}_\alpha^{uTs-t*}\cdot e_\beta^v\\ +\delta^{uv}\delta^{st}\gamma^{st}e_\alpha^u\cdot e_\beta^v\end{array} &
\begin{array}{c}\delta^{uv}\mu^v M_{\alpha\beta}^{vTRst}\\ +e_\alpha^u\cdot\bar{\Psi}_\beta^{s-vRt}\\ +\bar{Y}_\alpha^{uTs-t*}\cdot e_\beta^v\\ -\delta^{uv}\delta^{st}e_\alpha^u\cdot\bar{D}^{st}\times e_\beta^v\end{array} &
e_\alpha^u\cdot\bar{\Psi}_\ell^{s-vV} \\ \hline
\begin{array}{c}\delta^{uv}\mu^v M_{\alpha\beta}^{vRTst}\\ +e_\alpha^u\cdot\bar{Y}_\beta^{s-vTt}\\ +\bar{\Psi}_\alpha^{uRs-t*}\cdot e_\beta^v\\ +\delta^{uv}\delta^{st}e_\alpha^u\cdot\bar{D}^{st*}\times e_\beta^v\end{array} &
\begin{array}{c}\delta^{uv}\mu^v M_{\alpha\beta}^{vRRst}\\ +e_\alpha^u\cdot\bar{Y}_\beta^{s-vRt}\\ +\bar{Y}_\alpha^{uRs-t*}\cdot e_\beta^v\\ +\delta^{uv}\delta^{st}e_\alpha^u\cdot\bar{\bar{I}}^{st}\cdot e_\beta^v\end{array} &
e_\alpha^u\cdot\bar{Y}_\ell^{s-vV} \\ \hline
\bar{\Psi}_k^{uV-t*}\cdot e_\beta^v & \bar{Y}_k^{uV-t*}e_\beta^v & \delta^{uv}\mu^v M_{k\ell}^{vVV}
\end{array}\right] \qquad (34)
$$

$$
C_{mn}^{uXs-vYt}=\left[\begin{array}{c:c:c}
\begin{array}{c}\delta^{uv}\mu^v C_{\alpha\beta}^{vTTst}\\ +e_\alpha^u\cdot\bar{\Psi}_\beta^{s-vTt0}\\ +\bar{\Psi}_\alpha^{uTs-t*}\cdot e_\beta^{v0}\\ +\delta^{uv}\delta^{st}\gamma^{st}e_\alpha^u\cdot e_\beta^{v0}\end{array} &
\begin{array}{c}\delta^{uv}\mu^v C_{\alpha\beta}^{vTRst}\\ +e_\alpha^u\cdot\bar{\Psi}_\beta^{s-vRt0}\\ +\mathfrak{B}_{\alpha\beta}^{uTs-vt}\\ -\delta^{uv}\delta^{st}e_\alpha^u\cdot(\bar{D}^{st}\times e_\beta^v)^0\end{array} &
e_\alpha^u\cdot\bar{\Psi}_\ell^{s-vV0} \\ \hline
\begin{array}{c}\delta^{uv}\mu^v C_{\alpha\beta}^{vRTst}\\ +e_\alpha^u\cdot\bar{B}_\beta^{s-vTt}\\ +\bar{\Psi}_\alpha^{uTs-t*}\cdot e_\beta^{v0}\\ +\delta^{uv}\delta^{st}e_\alpha^u\cdot\bar{D}^{st*}\times e_\beta^{v0}\end{array} &
\begin{array}{c}\delta^{uv}\mu^v C_{\alpha\beta}^{vRRst}\\ +e_\alpha^u\cdot\bar{B}_\beta^{s-vRt}\\ +\mathfrak{B}_{\alpha\beta}^{uRs-vt}\\ +\delta^{uv}\delta^{st}e_\alpha^u\cdot\mathfrak{J}_\beta^{s-vt}\end{array} &
e_\alpha^u\cdot\bar{B}_\ell^{s-vV} \\ \hline
\bar{\Psi}_k^{uV-t*}\cdot e_\beta^{v0} & \mathfrak{B}_{k\beta}^{uV-vt} & \delta^{uv}\mu^v C_{k\ell}^{vVV}
\end{array}\right] \qquad (35)
$$

$$
Q_{mn}^{uXs-vYt}=\left[\begin{array}{c:c:c}
\begin{array}{c}\delta^{uv}\mu^v Q_{\alpha\beta}^{vTTst}\\ +e_\alpha^u\cdot\bar{\Psi}_\beta^{s-vTt\infty}\\ +\bar{\Psi}_\alpha^{uTs-t*}\cdot e_\beta^{v\infty}\\ +\delta^{uv}\delta^{st}\gamma^{st}e_\alpha^u\cdot e_\beta^{v\infty}\end{array} &
\begin{array}{c}\delta^{uv}\mu^v Q_{\alpha\beta}^{vTRst}\\ +e_\alpha^u\cdot\bar{\Psi}_\beta^{s-vRt\infty}\\ +\mathfrak{P}_{\alpha\beta}^{uTs-vt}\\ -\delta^{uv}\delta^{st}e_\alpha^u\cdot(\bar{D}^{st}\times e_\beta^v)^\infty\end{array} &
e_\alpha^u\cdot\bar{\Psi}_\ell^{s-vV\infty} \\ \hline
\begin{array}{c}\delta^{uv}\mu^v Q_{\alpha\beta}^{vRTst}\\ +e_\alpha^u\cdot\bar{P}_\beta^{s-vTt}\\ +\bar{\Psi}^{uRs-1*}\cdot e_\beta^{v\infty}\\ +\delta^{uv}\delta^{st}e_\alpha^u\cdot D^{st*}\times e_\beta^{v\infty}\end{array} &
\begin{array}{c}\delta^{uv}\mu^v Q_{\alpha\beta}^{vRRst}\\ +e_\alpha^u\cdot\bar{P}_\beta^{s-vRt}\\ +\mathfrak{P}_{\alpha\beta}^{uRt-vt}\\ +\delta^{uv}\delta^{st}e_\alpha^u\cdot\mathfrak{S}_\beta^{s-vt}\end{array} &
e_\alpha^u\cdot\bar{P}_\ell^{s-vV} \\ \hline
\bar{\Psi}_k^{uV-t*}\cdot e_\beta^{v\infty} & \mathfrak{P}_{k\beta}^{uV-vt} & \delta^{uv}\mu^v Q_{k\ell}^{vVV}
\end{array}\right] \qquad (36)
$$

where

$$\mathcal{B}_{m\beta}^{uXs-vt} = \bar{Y}_m^{uXs-t*} \cdot e_\beta^{v0} + \bar{B}_m^{uXs-t*} \cdot e_\beta^v$$

$$\mathcal{P}_m^{uXs-vt} = \bar{Y}_m^{uXs-t*} \cdot e_\beta^{v\infty} + 2\bar{B}_m^{uXs-t*} \cdot e_\beta^{v0} + \bar{P}_m^{uXs-t*} e_\beta^v$$

$$\bar{\mathfrak{J}}_\beta^{s-vt} = \bar{\bar{I}}^{st} \cdot e_\beta^{v0} + \bar{\bar{J}}^{st} \cdot e_\beta^v$$

$$\bar{\mathfrak{S}}_\beta^{s-vt} = \bar{\bar{I}}^{st} \cdot e_\beta^{v\infty} + 2\bar{\bar{J}}^{st} \cdot e_\beta^{v0} + \bar{\bar{S}}^{st} \cdot e_\beta^v \tag{37}$$

Resultant Control Forces and Torques

The resultant control forces and torques are

$$\bar{F}^c = \sum_{j=1}^{N} \sum_{i=1}^{N_j} \bar{F}^{jc(i)}$$

$$\Delta\bar{F}^c = \sum_{j=1}^{N} \sum_{i=1}^{N_j} \Delta\bar{F}^{jc(i)}$$

$$\bar{T}^{uc} = \sum_{j=u}^{N} \epsilon^{uj} \sum_{i=1}^{N_j} \bar{L}^{u(ji)} \times \bar{F}^{jc(i)}$$

$$\Delta\bar{T}^{uc} = \sum_{j=u}^{N} \epsilon^{uj} \sum_{i=1}^{N_j} \bar{L}^{u(ji)} \times \Delta\bar{F}^{jc(i)}$$

$$X_m^{uXs} = \eta^{us} \sum_{j=1}^{N} \epsilon^{uj} \sum_{i=1}^{N_j} \bar{\mathfrak{X}}_m^{uXs(ji)} \cdot \bar{F}^{jc(i)}, \tag{38}$$

Gravitational Effects

All of the gravitational effects in system dynamic equations are associated with the constant ζ. Due to page limits of this paper, we shall only list a few.

$$\bar{H}^{u1(G)} = \sum_{j=u}^{N} \epsilon^{uj}\mu^j [3e \times (\bar{\bar{A}}^j + \bar{d}^j\bar{d}^j) \cdot e + \bar{g}^{u(j)} \times \bar{\bar{\Omega}} \cdot \bar{g}^{1(j)}]$$

$$\begin{aligned}\bar{\bar{S}}^{uv(G)} = \sum_{j=u,v}^{N} & \epsilon^{uj}\epsilon^{vj}\mu^j [\bar{\bar{I}}^j + 3e \cdot \bar{\bar{A}}^j \cdot e + 3(e \cdot \bar{\bar{A}}^j \cdot e)\bar{\bar{\delta}} \\ & -3\bar{\bar{A}}^j \cdot (ee) - \bar{g}^{u(j)} \times \bar{\bar{\Omega}} \cdot (\bar{g}^{v(j)}) \\ & + \bar{d}^j \times \bar{\bar{\Omega}} \cdot (\bar{d}^j) - (\bar{d}^j \cdot \bar{\bar{\Omega}}) \\ & \times (\bar{d}^j) + (\bar{R}^j + \bar{d}^j) \times \bar{\bar{\Omega}} \cdot (\bar{g}^{v(j)})]\end{aligned} \tag{39}$$

Computer Program Development

A general-purpose multibody dynamics and control simulation computer program, called ALLFLEX, was developed from the above analysis. The program structure and computational logic are shown in Fig. 5. A central executive subroutine regulates the execution of second-level subroutines which have the following functions.

1) Simulation input data include: system topology; hinge-to-hinge vectors and mounting matrices; physical data (mass/mass center/inertia) of individual bodies; flexible structural modal data; degrees of freedom of gimbal rotation; orbital dynamics information; and state variables initial conditions.

2) Calculations of time-independent rigid and flexible body mass properties of individual bodies are off-line computations which are the main interface with structural dynamics programs (SPAR and NEPSEP). The resulting compressed modal data have much smaller dimensions and the on-line computer storage space is minimized.

3) Because of large-angle articulation between bodies and the dynamic elastic deformation within bodies, the system kinematic relations, such as the coordinate and gimbal transformation matrices, angular velocity and acceleration of individual bodies, and direct position vectors and derivatives are all time-dependent by nature. They are updated by the feedback of nominal and perturbed state variables.

4) The angular velocity and acceleration of individual bodies will transform the time-independent mass properties into time-dependent mass properties. This information, together with the system kinematic relations, will determine the overall system properties.

5) Matrix formulation for the system dynamic equations of nominal motion and perturbed motion are coded separately. Integration steps for nominal state variables and perturbed state variables can vary because of the difference in dynamic behavior. The nominal state variables are also required by the perturbed dynamic equations to update the coefficient matrices.

6) System nominal motion and perturbed motion are combined to form the system total motion. Program output includes the translation and rotation of each body and the dynamic elastic deformations at any specified location in the bodies. Output station modal data are needed for these calculations.

7) Provisions were made to add the nominal and perturbed control forces and torques for system feedback control.

8) Coefficient matrices of the linearized system perturbation equations will be inputs to the generalized stability analysis program (GSA).

Discussion and Conclusion

The complexity of this multibody problem comes from two sources. The first one is the system dynamic equilibrium. The time-independent and dependent rigid and flexible body properties defined in its moving body coordinate frame must be transformed into the system properties through rigid and elastokinematic relations. When they are written in the matrix formulation for computer program development, proper insertions of coordinate transformation matrices, which are nonlinear trigonometric functions of mounting and gimbal rotational angles, must be handled with extreme care. Derivation of system dynamic equations should be kept at the topological tree level. Inserting sines and cosines in the analysis will not only lengthen the equations but also cloud the issue. The second source comes from the system topology. All quantities in the analysis are defined carefully with unique superscripts and subscripts. The superscripts for defining the interbody relations and various interactions among quasistatic motion and natural vibration are playing different levels of tensorial interrelations from the subscripts, which represent the components of dynamic vectors and tensors as well as modal numbers. Proper definitions of these quantities are essential to the success of general-purpose program development. Any simplification may reduce the generality of our simulation capability and it is really defeating the purpose of understanding this complex problem.

The system dynamic equations have represented the complete coupling among rigid-body translation and rotation, the quasistatic translation and rotation, and also natural vibration. The coupling scheme is based on the super-parallel axis theorem. The mass center, inertia, and modal effects for each body are first calculated with respect to the origin (i.e., the hinge point) of the body coordinates. Then, with the help from incidence matrix and limb-branch matrix, the direct position vectors are able to transform the dynamic effect of one body to another location in the system. In addition to the system stiffness and damping (i.e., K's and V's), the nominal motion is also influencing the stiffness and damping terms of the linearized perturbed equations. These effects are called the dynamic stiffness and damping. When we freeze the time, we may determine all the poles and zeros in the system. Large-angle articulation will cause the changes in system characteristics and, thus, changes of all the locations of poles and zeros. Obviously the more violent nominal motion or drastic change of direction of motion will excite more perturbed motion.

In conclusion, this analysis and simulation computer program have provided us with a useful tool to study all of the effects mentioned above, through which we will be able to design a better control system to meet the fast maneuvering and fine-pointing requirements. In the future development of large flexible space structures, the system properties such as mass center, inertias, flexibilities, etc., at each stage of construction will have different dynamic behavior and thus requires different control design. This simulation capability is very versatile in making changes of system topology. Therefore, it is useful to the iterative design studies and system improvement.

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